SYMPLECTIC FIBRATIONS AND RIEMANN-ROCH NUMBERS OF REDUCED SPACES

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Abstract. In this article we give formulas for the Riemann-Roch number of a symplectic quotient arising as the reduced space of a coadjoint orbit $\mathcal{O}_\Lambda$ (for $\Lambda \in \mathfrak{g}^*$ close to 0) as an evaluation of cohomology classes over the reduced space at 0. This formula exhibits the dependence of the Riemann-Roch number on Λ. We also express the formula as a sum over the components of the fixed point set of the maximal torus. Our proof applies to Hamiltonian G-manifolds even if they do not have a compatible Kähler structure, using the definition of quantisation in terms of the Spin-$\mathbb{C}$ Dirac operator.

1. Introduction

Let (M, ω) be a compact symplectic manifold possessing a Hamiltonian action of a compact connected simply connected Lie group G, with moment map $\mu : M \to \mathfrak{g}^*$ (where $\mathfrak{g}$ is the Lie algebra of G). One can form the symplectic reduction

$$M_0 = \mu^{-1}(0)/G,$$

or more generally

$$M_\Lambda = \mu^{-1}(\mathcal{O}_\Lambda)/G$$

for $\Lambda \in \mathfrak{g}$, where $\mathcal{O}_\Lambda \subseteq \mathfrak{g}^*$ is the orbit of Λ under the coadjoint action.

We assume 0 is a regular value of μ, and 0 is a regular value of $\mu_\Lambda : M \times \mathcal{O}_\Lambda \to \mathfrak{g}^*$ where $\mu_\Lambda(m, \xi) = \mu(m) - \xi$ for $m \in M$ and $\xi \in \mathcal{O}_\Lambda$. This is equivalent to assuming that G acts with finite stabilizers on $\mu^{-1}(0)$ (resp. $\mu_\Lambda^{-1}(0)$) [8], so under this hypothesis M_0 and M_Λ have at worst finite quotient singularities. We assume that G acts freely on $\mu_\Lambda^{-1}(0)$ and $\mu^{-1}(0)$, so that M_Λ is a smooth symplectic manifold. Denote the symplectic form on M_Λ by ω_Λ.

Let L be a complex line bundle over M_Λ with a connection whose curvature is equal to ω_Λ, called a *prequantum line bundle.* If M has a complex structure compatible with the symplectic structure (in other words M is Kähler), then the *quantisation* of M_Λ is defined as the virtual vector space

$$\mathcal{Q}(L^k) = H^0(M, L^k) - H^1(M, L^k) + H^2(M, L^k) - \dots \tag{1}$$

where $H^j(M, L^k)$ is the j-th Dolbeault cohomology of M with coefficients in L^k. (When k is very large, all the $H^j(M, L^k) = 0$ for $j \neq 0$, so the quantisation $\mathcal{Q}$ is simply a vector

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space.) The dimension of the quantisation is given by the Riemann-Roch number. The formula for the Riemann-Roch number in terms of characteristic classes is given below at (3).

As has been observed by Duistermaat [5], Guillemin [7] and Vergne [21], even when M is not Kähler (but is equipped only with an almost complex structure compatible with the symplectic structure – such almost complex structures always exist, as explained in the above references) one can still define the quantisation using an elliptic complex given by the Spin-$\mathbb{C}$ Dirac operator. In this more general situation, the dimension of the quantisation is still given by the Riemann-Roch number, the formula for which is still given in terms of characteristic classes by (3) below (see [5], Proposition 13.1). The quantisation using the spin-$\mathbb{C}$ Dirac operator has been extensively studied by Meinrenken [17, 18].

In this article we give formulas for the Riemann-Roch number of L^k over the reduced space M_Λ (for $\Lambda \in \mathfrak{g}^*$ close to 0) as an evaluation of cohomology classes over the reduced space M_0. This formula exhibits the dependence of the Riemann-Roch number on Λ. We also express the formula as a sum over the components of the fixed point set of the maximal torus. In Section 2.1 we give a simple proof for the Kähler case, which was suggested by the referee. In Section 2.2 we treat the non-Kähler case.

2. Symplectic fibrations

It is a standard result (see for example [8]) that for Λ in a neighbourhood of 0 in $\mathfrak{g}^*$, we have a fibration

$$\begin{array}{ccc} \mathcal{O}_\Lambda & \longrightarrow & M_\Lambda \\ & & \big\downarrow \pi \\ & & M_0 \end{array} \tag{2}$$

If M is Kähler and the G action preserves the Kähler structure, then M_Λ and M_0 are also Kähler and (2) is a fibration of Kähler manifolds.

Let L be a line bundle over M_Λ with Chern character equal to e^{ω_Λ}, (for example a prequantum line bundle) and let $k \in \mathbb{Z}$. The *Riemann-Roch number* of L^k is then

$$RR(M_\Lambda, L^k) = \int_{M_\Lambda} \mathrm{ch}(L^k)\,\mathrm{Td}(M_\Lambda) \tag{3}$$

where $\mathrm{Td}(M_\Lambda)$ means $\mathrm{Td}(TM_\Lambda)$. The goal of this section is to express the Riemann-Roch number (3) as far as possible using terms defined on M_0.

Let

$$\lambda = k\Lambda, \tag{4}$$

where k is a positive integer. We require that λ lie in the weight lattice $\Lambda^W \subset \mathfrak{t}^*$, which is the dual of the integer lattice $\Lambda^I \subset \mathfrak{t}$ (the kernel of the exponential map $\mathfrak{t} \to T$). We do not require that $\Lambda \in \Lambda^W$.

2.1. **The Kähler case.** When M_0 and M_Λ are Kähler, there is the following straightforward proof (which was pointed out by the referee). Let $V(\lambda)^*$ be the irreducible representation of G with lowest weight $-\lambda$ (the dual of the irreducible representation with highest weight λ). By using the principal G-bundle $p_0 : \mu^{-1}(0) \to M_0$, this representation yields a vector bundle $\mathcal{V}(\lambda)^*$ on M_0. We introduce a line bundle L_0 on M_0 for which $c_1(L_0) = [\omega_0]$, where ω_0 is the Kähler form of M_0 and $[\omega_0]$ is its de Rham cohomology class.

If we let G_Λ denote the stabilizer of Λ under the adjoint action, then

$$L^k \cong (\pi^* L_0^k) \otimes \mathcal{L}_{-\lambda} \tag{5}$$

where $\mathcal{L}_{-\lambda}$ is the complex line bundle associated with the principal G_Λ-bundle

$$p_\Lambda : \mu^{-1}(\Lambda) \to \mu^{-1}(\Lambda)/G_\Lambda \cong M_\Lambda$$

and with the complex representation of G_Λ of dimension 1 and weight $-\lambda$. This is true because the first Chern class of L_k is equal to $L = \pi^*[\omega_0] + [\widetilde{\Omega}_\Lambda]$, where $\widetilde{\Omega}_\Lambda$ is a form on M_Λ which, when restricted to a fiber, is the Kirillov-Kostant-Souriau form Ω_Λ on the coadjoint orbit $\mathcal{O}_\Lambda$. Furthermore,

$$\mu^{-1}(\Lambda)/G_\Lambda \cong \mu^{-1}(0)/G_\Lambda$$

for Λ close to 0. This yields the fibration π given in (2). By the Borel-Weil-Bott theorem [3], the pushforward of $\mathcal{L}_{-\lambda}$ under this fibration is the vector bundle $\mathcal{V}(\lambda)^*$, and all higher direct images vanish.

By the Grothendieck-Riemann-Roch theorem [10], we have

$$RR(M_\Lambda, L^k) = RR(M_0, L_0^k \otimes \mathcal{V}(\lambda)^*) \tag{6}$$

using (5) and the fact that higher direct images vanish.

2.2. **The non-Kähler case.** When M_0 and M_Λ are not Kähler, we must rely on an explicit argument using the Chern character and the Todd class. By the Normal Form Theorem ([8], Theorem 39.3 and Prop. 40.1) we can write the symplectic form on M_Λ as $\omega_\Lambda = \pi^*\omega_0 + \widetilde{\Omega}_\Lambda$, where $\widetilde{\Omega}_\Lambda$ is a form on M_Λ which, when restricted to a fiber, is the Kirillov-Kostant-Souriau form Ω_Λ on the coadjoint orbit $\mathcal{O}_\Lambda$. Thus

$$\mathrm{ch}(L^k) = e^{k\omega_\Lambda} = e^{k\pi^*\omega_0} e^{k\widetilde{\Omega}_\Lambda}.$$

Next, we can split the tangent bundle of M_Λ as $TM_\Lambda = \pi^*TM_0 \oplus \mathcal{T}$, where $\mathcal{T}$ is the vertical bundle whose fiber over a point $x \in M_\Lambda$ is the tangent space to the fiber of π over x. Since the Todd class is multiplicative, $\mathrm{Td}(M_\Lambda) = \mathrm{Td}(TM_\Lambda) = \pi^* \mathrm{Td}(TM_0)\,\mathrm{Td}(\mathcal{T})$.

Combining with the expression for $\mathrm{ch}(L^k)$, we have

$$RR(M_\Lambda, L^k) = \int_{M_\Lambda} \mathrm{ch}(L^k)\,\mathrm{Td}(M_\Lambda) = \int_{M_\Lambda} e^{k\pi^*\omega_0}\,\pi^*\mathrm{Td}(M_0) e^{k\widetilde{\Omega}_\Lambda}\,\mathrm{Td}(\mathcal{T}). \tag{7}$$

The product of the first two factors $e^{k\pi^*\omega_0}\pi^*\,\mathrm{Td}(M_0)$ is written in terms of objects defined solely on the base M_0, and so we turn our attention to the other factors $e^{k\widetilde{\Omega}_\Lambda}\,\mathrm{Td}(\mathcal{T})$. Our strategy will be to integrate over the fiber of π, and be left with an integral over only the

base M_0. Now $\mathcal{T}$ is a bundle over M_Λ, and so $\mathrm{Td}(\mathcal{T}) \in H^*(M_\Lambda)$. If $\imath_x \colon \mathcal{O}_\Lambda \hookrightarrow M_\Lambda$ is the inclusion map from $\mathcal{O}_\Lambda$ to the fiber over $x \in M_0$, then $\imath_x^*(\mathcal{T}) \cong T\mathcal{O}_\Lambda$.

Suppose M is equipped with a complex vector bundle $\mathcal{V}$ (with fiber $\mathfrak{t}\otimes\mathbb{C}$) with an action of G compatible with the action on M. Then $\mathcal{V}$ descends to a vector bundle E on M_0. The characteristic classes of E come from the invariant polynomials on $\mathfrak{g}$ via the Kirwan map. (The Kirwan map $\kappa : H^*_G(M) \to H^*(M_0)$ is the composition of the restriction map $r : H^*_G(M) \to H^*_G(\mu^{-1}(0))$ with the isomorphism $H^*_G(\mu^{-1}(0)) \cong H^*(\mu^{-1}(0)/G)$, which is valid when 0 is a regular value for μ. See [12].) We assume cohomology with rational, real or complex coefficients.

We assume the vector bundle E over M_0 has the property that its pullback to M_Λ splits as the direct sum of a collection of line bundles L_i (in other words M_Λ is a splitting manifold for E over M_0). We then define $e_i \in H^2(M_\Lambda)$ by $e_i = c_1(L_i)$. The characteristic class of E associated to an invariant polynomial $\tau \in S(\mathfrak{t}^*)^W$ is then given by

$$c_\tau(E) = \tau(e_1, \ldots, e_\ell)$$

(where ℓ is the rank of T). For example, if $G = U(n)$ the invariant polynomials are generated by the elementary symmetric polynomials [19]. The motivating example (the case treated in [13]) is the case where M_0 is the moduli space $M(n,d)$ of semistable holomorphic vector bundles of rank n and degree d over a Riemann surface (when n and d are two coprime positive integers), and M_Λ is the corresponding moduli space of parabolic bundles. In this case the vector bundle E is the universal bundle (see [1]).

By Section 14 in [11],

$$\mathrm{Td}(\mathcal{O}_\Lambda) = \prod_{\gamma>0} \frac{\gamma(\mathcal{E})}{1 - e^{-\gamma(\mathcal{E})}} = \prod_{\gamma>0} \frac{\gamma(\mathcal{E}) e^{\frac{1}{2}\gamma(\mathcal{E})}}{(e^{\frac{1}{2}\gamma(\mathcal{E})} - e^{-\frac{1}{2}\gamma(\mathcal{E})})}$$

where γ are the roots of G, and $\mathcal{E} = (e_1, \ldots, e_\ell) \in H^2(M_\Lambda) \otimes \mathbb{R}^\ell$.

For example, if $G = U(n)$, under these hypotheses we obtain that the j-th Chern class is

$$c_j(E) = \kappa(\{\tau_j\})$$

where τ_j (the j-th elementary symmetric polynomial) is regarded as an element of $H^*_G(\mathrm{pt}) = S(\mathfrak{g}^*)^G$ and $\kappa : H^*_G(M) \to H^*(M_0)$ is the Kirwan map.

We introduce a basis $\hat{u}_i, i = 1, \ldots, \ell$ for the integer lattice Λ^I of G (where ℓ is the rank of G). This enables us to define elements $e_j \in H^2(M_\Lambda), j = 1, \ldots, \ell$ satisfying $c_1(L_j) = e_j$, where e_j restricts on the fibers of π to the generator α_j (for $j = 1, \ldots, \ell$) of $H^2(G/T, \mathbb{Z}) \cong H^1(T, \mathbb{Z})$ corresponding to the j-th fundamental weight of G (an element of $\mathrm{Hom}(T, U(1)) \cong H^1(T, \mathbb{Z})$)). Using this notation, we have

Lemma 2.1. *Let $\Lambda = \sum_{i=1}^{\ell} \Lambda_i \hat{u}_i$. Then the standard Kirillov-Kostant symplectic form Ω_Λ on $\mathcal{O}_\Lambda$ is given by*

$$\Omega_\Lambda = \sum_{j=1}^{\ell} \Lambda_j \alpha_j,$$

where the $\hat{u}_i$ and α_j are as defined above.

Proof: This is a standard result (see for instance [2], Lemma 7.22). $\square$

The roots γ also lie in the weight lattice Λ^W. Writing the pairing of $\mathfrak{t}^*$ and $\mathfrak{t}$ as as $(\cdot,\cdot)$ we have (still when restricted to the fiber)

$$\begin{aligned} e^{k\widetilde{\Omega}_\Lambda}\,\mathrm{Td}(\mathcal{O}_\Lambda) &= e^{\sum \lambda_i e_i}\prod_{\gamma>0} e^{\frac{1}{2}(\gamma,\mathcal{E})}\left[\prod_{\gamma>0}\frac{(\gamma,\mathcal{E})}{(e^{\frac{1}{2}(\gamma,\mathcal{E})}-e^{-\frac{1}{2}(\gamma,\mathcal{E})})}\right] \\ &= e^{(\lambda,\mathcal{E})}e^{\frac{1}{2}\sum_{\gamma>0}(\gamma,\mathcal{E})}\prod_{\gamma>0}\frac{(\gamma,\mathcal{E})}{(e^{\frac{1}{2}(\gamma,\mathcal{E})}-e^{-\frac{1}{2}(\gamma,\mathcal{E})})} \\ &= e^{(\lambda+\rho,\mathcal{E})}\prod_{\gamma>0}\frac{(\gamma,\mathcal{E})}{(e^{\frac{1}{2}(\gamma,\mathcal{E})}-e^{-\frac{1}{2}(\gamma,\mathcal{E})})} \qquad (8)\end{aligned}$$

where ρ is half the sum of the positive roots. Notice that

$$e^{k\widetilde{\Omega}_\Lambda}\,\mathrm{Td}(\mathcal{O}_\Lambda)[\mathcal{O}_\Lambda] = RR(\mathcal{O}_\Lambda, L^k) \qquad (9)$$

which equals $\dim V_\lambda$ by the Bott-Borel-Weil theorem (see [3] or [20]), where V_λ is a representation of G with highest weight λ (we have assumed that λ is in the fundamental Weyl chamber).

After evaluating on the fundamental cycle of M_Λ, the equation (8) equals

$$\frac{1}{|W|}\left(\sum_{\sigma\in W}(-1)^\sigma\frac{e^{(\sigma(\lambda+\rho),\mathcal{E})}}{\prod_{\gamma>0}(e^{\frac{1}{2}(\gamma,\mathcal{E})}-e^{-\frac{1}{2}(\gamma,\mathcal{E})})}\right)\prod_{\gamma>0}(\gamma,\mathcal{E})[M_\Lambda]. \qquad (10)$$

The expression in brackets in (10) is unchanged under the action of the Weyl group $\mathcal{E}\mapsto w\mathcal{E}$.

Let $\{\tau_r\}$ $(r=1,\dots,n(G))$ be a set of generators for the ring $S(\mathfrak{t}^*)^W$ of Weyl invariant polynomials on $\mathfrak{t}$, where $n(G)$ is the number of generators. By Proposition 3.6 in [13], $\tau_r(e_1,\dots,e_\ell)) = \pi^* a_r$ for a_r a class on M_0. Therefore the factor (10) can be written as a function of the invariant polynomials τ_r applied to the e_j's,

$$\mathcal{S}_\lambda\big(\tau_r(e_1,\dots,e_\ell)\big) \stackrel{\mathrm{def}}{=} \sum_{\sigma\in W}(-1)^\sigma\frac{e^{(\sigma(\lambda+\rho),\mathcal{E})}}{\prod_{\gamma>0}(e^{\frac{1}{2}(\gamma,\mathcal{E})}-e^{-\frac{1}{2}(\gamma,\mathcal{E})})} \qquad (11)$$

We see that $\mathcal{S}_\lambda$ is actually a polynomial in the e_j. Because $e_j^{N+1}=0$ where $2N=\dim_{\mathbb{R}} M_\Lambda$, it follows immediately that after evaluating on the fundamental class of M_0,

Theorem 2.2. *$\mathcal{S}_\lambda$ is a polynomial in λ of degree $\leq N$.*

We can replace the term $e^{k\widetilde{\Omega}_\Lambda}\,\mathrm{Td}(V)$ in the integral with

$$\frac{1}{|W|}\mathcal{S}_\lambda(\pi^*a_1,\dots,\pi^*a_{n(G)})\prod_{\gamma>0}(\gamma,\mathcal{E})$$

to get

$$RR(M_\Lambda, L^k) = \frac{1}{|W|}\int_{M_\Lambda} e^{k\pi^*\omega_0}\pi^*\,\mathrm{Td}(M_0)\mathcal{S}_\lambda(\pi^*a_1,\dots,\pi^*a_{n(G)})\prod_{\gamma>0}(\gamma,\mathcal{E}). \qquad (12)$$

All of the factors in the integral except for $\prod(\gamma,\mathcal{E})$ are constant on each fiber, and so we have

$$RR(M_\Lambda, L^k) = \frac{1}{|W|}\int_{M_0} e^{k\omega_0}\,\mathrm{Td}(M_0)\mathcal{S}_\lambda(a_1,\ldots,a_{n(G)})\int_{\mathcal{O}_\Lambda}\prod_{\gamma>0}(\gamma,\mathcal{E}). \tag{13}$$

Since

$$\int_{\mathcal{O}_\Lambda}\prod_{\gamma>0}(\gamma,\mathcal{E}) = |W|,$$

we have finally

Theorem 2.3. *The Riemann-Roch number of a symplectic fibration M_Λ is given by*

$$RR(M_\Lambda, L^k) = \int_{M_0} e^{k\omega_0}\,\mathrm{Td}(M_0)\mathcal{S}_\lambda(a_1,\ldots,a_{n(G)}). \tag{14}$$

where $\mathcal{S}_\lambda$ is defined at (11).

Theorem 2.4. *When $\Lambda\in\Lambda^W$ is a weight in the fundamental Weyl chamber, then the limit as $k\to\infty$ of*

$$\frac{RR(M_\Lambda, L^k)}{k^N}$$

(where as above $2N=\dim_{\mathbb{R}}(M_\Lambda)$) is $\mathrm{vol}\,M_0\dim V_{\Lambda-\rho}$.

Proof: This limit is given by $\mathrm{vol}\,M_\Lambda$. The symplectic volume is given by $\mathrm{vol}\,M_0\,\mathrm{vol}\,\mathcal{O}_\Lambda$. The symplectic volume of $\mathcal{O}_\Lambda$ is given at [2] (Proposition 7.26) as

$$\mathrm{vol}(\mathcal{O}_\Lambda) = \frac{\prod_{\alpha>0}<\alpha,\Lambda>}{\prod_{\alpha>0}<\alpha,\rho>}. \tag{15}$$

This gives the value of $\dim V_{\Lambda-\rho}$, using the Weyl dimension formula [6]. □

Proposition 2.5. *Let $\lambda\in\Lambda^W$ be a weight in the fundamental Weyl chamber, and define $\Lambda(k)=\lambda/k$. Let $N_0=\frac{1}{2}\mathrm{dim}M_0$. Then*

$$\lim_{k\to\infty}\frac{1}{k^{N_0}}RR(M_\Lambda, L^k) = (\mathrm{vol}M_0)(\dim V_{\lambda-\rho}).$$

Proof: For $X\in\mathbf{t}$ we introduce

$$S_\lambda\big(X\big) \stackrel{\mathrm{def}}{=} \sum_{\sigma\in W}(-1)^\sigma\frac{e^{(\sigma(\lambda+\rho),X)}}{\prod_{\gamma>0}(e^{\frac{1}{2}(\gamma,X)}-e^{-\frac{1}{2}(\gamma,X)})} \tag{16}$$

Notice that we are fixing λ and allowing $\Lambda=\lambda/k$ to vary as k varies. Notice that by the Weyl character formula ([20], Proposition 14.2.2) we have

$$S_\lambda(X)=\chi_\lambda(\exp X)$$

where χ_λ is the character of the representation with lowest weight $-\lambda$. Theorem 2.3 gives the result, noting that in the limit $k\to\infty$ the leading order term in k comes by integrating $(k\omega_0)^{N_0}$ so the factor $\mathcal{S}_\lambda(a_1,\ldots,a_{n(G)})$ contributes only its value when all the arguments a_i are replaced by 0, in other words when the argument of $S_\lambda(X)$ is replaced by $X=0$. This is the value $\chi_\lambda(0)$, in other words the dimension of $V_{\lambda-\rho}$. □

Example 2.6. When $G = SU(2)$, the value of $\mathcal{S}_\lambda$ is (recalling that λ is a positive integer)

$$\mathcal{S}_\lambda(a_2) = \kappa(S_\lambda(X))$$

where

$$S_\lambda(X) = \frac{1}{2}\frac{e^{(\lambda+1)X} - e^{-(\lambda+1)X}}{e^X - e^{-X}} \tag{17}$$

$$= \frac{1}{2}(\cosh(X) + \cdots + \cosh(\lambda-1)X + \cosh \lambda X). \tag{18}$$

Here we have introduced a formal variable X for which $\kappa(X^2) = a_2 \in H^4(M_0)$. It follows that $\mathcal{S}_\lambda$ is a polynomial in λ of order $N+1$, because the terms which contribute from the Taylor expansion of order X^N are $\sum_{j=1}^{\lambda} j^N$ which is of order λ^{N+1}. Because we are integrating over M_Λ, the ring of polynomials in the variable X gets truncated by imposing the relation $X^{N+1} = 0$. In this example $S_\lambda(0) = \lambda/2$.

3. The Jeffrey-Kirwan residue formula

In the final two sections of this paper, we express the Riemann-Roch number on a symplectic fibration in terms of data at the fixed point set of the maximal torus T of G (assuming Λ is generic so its stabilizer under the coadjoint action is T). In this way, we obtain a second proof of Theorem 2.3.

The residue formula [14] expresses cohomology pairings on reduced spaces M_0 in terms of a multi-dimensional residue of certain rational holomorphic functions on $\mathfrak{t}$ [14]. This formula is valid provided 0 is a regular value of the moment map. The cohomology classes β_0 on M_0 are assumed to come from equivariant cohomology classes β on M via the Kirwan map [12]. The fixed point data are

- the value of the moment map at a component F of the fixed point set of the maximal torus T
- the restriction of β to the F
- the equivariant Euler class e_F of the normal bundle to F (which involves the weights of the action of T on the normal bundle, as well as the ordinary Chern roots of the normal bundle).

The residue formula takes the form

$$\int_{M_0} e^{\omega_0}\beta_0 = C\mathrm{Res}\left(\sum_F \int_F \frac{e^{\omega+\mu(F)(X)}\beta(X)}{e_F(X)}\right) \tag{19}$$

where C is a nonzero constant, $X \in \mathfrak{t} \otimes \mathbb{C}$ is a formal variable (in the Cartan model for equivariant cohomology), and Res is defined at (19) below.

We can readily identify the equivariant cohomology class giving rise to the Riemann-Roch number of a prequantum line bundle. The class e^{ω_0} is the Chern character of the line bundle over M_0, while $e^{\omega+\mu(F)(\cdot)}$ is the equivariant Chern character of the line bundle over M. The relevant class β_0 is the Todd class of M_0, which arises in the image of the Kirwan map using the equivariant Todd class Td_G (see [16] and Proposition 4.1 below). The residue formula has been applied to studying Riemann-Roch numbers in [15] and

[16]. Here we study the residue formula for the Riemann-Roch number on a symplectic fibration.

The residue formula applies to compact M reduced by any compact group G. The computation of terms in the residue formula depends on the choice of a cone Γ in $\mathfrak{t}$, even though the result of the formula is independent of this choice. Let $\gamma_1, \ldots, \gamma_k$ be the set of all weights that occur by the T action at any of the fixed point components. Choose some $\xi \in \mathfrak{t}$ such that $\gamma_i(\xi) \neq 0$ for all i. Let $\beta_i = \gamma_i$ if $\gamma_i(\xi) > 0$ and $\beta_i = -\gamma_i$ if $\gamma_i(\xi) < 0$. Thus $\beta_i(\xi) > 0$ for all i. The cone Γ is the set of all vectors in $\mathfrak{t}$ which behave like ξ:

$$\Gamma = \{X \in \mathfrak{t} : \ \beta_i(X) > 0, \text{ for all } i\}.$$

Theorem 3.1 (Jeffrey-Kirwan)**.** *Let (M, ω) be a compact symplectic manifold with a Hamiltonian T action and moment map Φ, where T is a compact torus. Denote by $\mathcal{F}$ the connected components of the fixed point set of T on M. Let p be a regular value of Φ and ω_p the Marsden-Weinstein reduced symplectic form on M_p. Then for $\beta \in H_T^*(M)$ and $\kappa_p : H_T^*(M) \to H^*(M_p)$ we have*

$$\int_{M_p} \kappa_p(\beta) e^{\omega_p} = C \cdot \mathrm{res}^\Gamma \left(\sum_{F \in \mathcal{F}} e^{i(\Phi(F)-p)(X)} \int_F \frac{\iota_F^*(\beta(X) e^{\omega})}{e_F(X)} [dX] \right) \tag{20}$$

where C is a non-zero constant, X is a variable in $\mathfrak{t} \otimes \mathbb{C}$, and $e_F(X)$ is the equivariant Euler class of the normal bundle to F in M. The multi-dimensional residue res^Γ *is defined below at (21).*

The residue can be defined as follows (see [16] Proposition 3.4). For f a meromorphic function of one complex variable z which is of the form $f(z) = g(z)e^{i\lambda z}$ where g is a rational function, we define

$$\mathrm{res}_z^+ \, f(z)dz = \sum_{b \in \mathbb{C}} \mathrm{res}(g(z)e^{i\lambda z}; z = b).$$

We extend this definition by linearity to linear combinations of functions of this form.

Viewing f as a meromorphic function on the Riemann sphere and observing that the sum of all the residues of a meromorphic 1-form on the Riemann sphere is 0, we observe that

$$\mathrm{res}_z^+ \left(f(z)dz\right) = -\mathrm{Res}_{z=\infty} \left(f(z)dz\right).$$

If $X \in \mathfrak{t}$, define

$$h(X) = \frac{q(X)e^{i\lambda(X)}}{\prod_{j=1}^k \beta_j(X)}$$

for some polynomial function $q(X)$ of X and some $\lambda, \beta_1, \ldots, \beta_k \in \mathfrak{t}^*$. Suppose that λ is not in any proper subspace of $\mathfrak{t}^*$ spanned by a subset of $\{\beta_1, \ldots, \beta_k\}$. Let Γ be any nonempty open cone in $\mathfrak{t}$ contained in some connected component of

$$\{X \in \mathfrak{t} : \beta_j(X) \neq 0, 1 \leq j \leq k\}.$$

Then for a generic choice of coordinate system $X = (X_1, \ldots, X_l)$ on $\mathfrak{t}$ for which $(0, \ldots, 0, 1) \in \Gamma$ we have

$$\mathrm{res}^\Gamma(h(X)[dX]) = \mathrm{Jac}\, \mathrm{res}_{X_1}^+ \circ \ldots \circ \mathrm{res}_{X_l}^+ \left(h(X)dX_1 \ldots dX_l\right) \tag{21}$$

where the variables $X_1, \ldots, X_{m-1}$ are held constant while calculating $\mathrm{res}^+_{X_m}$, and Jac is the determinant of any $l \times l$ matrix whose columns are the coordinates of an orthonormal basis of $\mathfrak{t}$ defining the same orientation as the chosen coordinate system. We assume that if $(X_1, \ldots, X_l)$ is a coordinate system for $X \in \mathfrak{t}$, then $(0, 0, \ldots, 1) \in \Gamma$. We also require an additional technical condition on the coordinate systems, which is valid for almost any choice of coordinate system (see Remark 3.5 (1) from [16]).

4. Fixed point formulas

We wish to use the residue formula to calculate the integral in (13). In (20) we are interested in $\beta(X) = \mathrm{Td}_G(M \times \mathcal{O}_\Lambda)\,\mathrm{Td}_G^{-1}(\mathfrak{g}_{\mathrm{ad}} \oplus \mathfrak{g}^*_{\mathrm{ad}})$ which maps to $\mathrm{Td}(M_\Lambda)$ under the Kirwan map.

The components of the fixed point set for the action of T on $M \times \mathcal{O}_\Lambda$ are of the form $F \times \{\sigma\Lambda\}$ where F is a component of the T fixed point set of M and $\sigma \in W$. The equivariant Euler class at this fixed point is $e_F(X)(-1)^\sigma \prod_{\gamma>0} \gamma(X)$. Thus the residue formula becomes

$$\int_{M_\Lambda} e^{k\omega_\Lambda}\,\mathrm{Td}(M_\Lambda) = C\mathrm{Res}\Bigg(\sum_{\sigma\in W}\sum_{F\in\mathcal{F}} (-1)^\sigma e^{\sigma\lambda(X)} e^{k\mu_F(X)} \times$$

$$\times \frac{\mathcal{D}^2(X)}{\prod_{\gamma>0}\gamma(X)\,\mathrm{Td}_G(\mathfrak{g}_{\mathrm{ad}} \oplus \mathfrak{g}^*_{\mathrm{ad}})} \int_{F\times\{\sigma\Lambda\}} \frac{e^{k\omega}}{e_F(X)} \mathrm{Td}_G(M)\,\mathrm{Td}_G(\mathcal{O}_\Lambda)\Bigg). \tag{22}$$

Here, ω is the symplectic form on F, C is an overall constant given in the statement of Theorem 3.1, $\mathcal{D}^2(X)$ is the product of all the roots of $\mathfrak{g}$, and $e_F(X) \in H^*_T(M)$ is the equivariant Euler class of the normal bundle to F. The F are components of the fixed point set $\mathcal{F}$ for the action of T on M. We use Proposition 4.1 below for the Todd class.

Proposition 4.1. *([15], Proposition 2.1) The formal equivariant cohomology class*

$$\mathrm{Td}_G(M)\,\mathrm{Td}_G^{-1}(\mathfrak{g}_{ad} \oplus \mathfrak{g}^*_{ad}) \tag{23}$$

maps to $\mathrm{Td}(M_0)$ under κ, where $\mathfrak{g}_{ad}$ (resp. $\mathfrak{g}^*_{ad}$) denotes the product bundle $M \times \mathfrak{g}$ (resp. $M \times \mathfrak{g}^*$) with G acting on $\mathfrak{g}$ by the adjoint action (resp. the coadjoint action).

Proof: We observe that $\kappa\big(\mathrm{Td}_G(M)\,\mathrm{Td}_G^{-1}(\mathfrak{g}_{\mathrm{ad}} \oplus \mathfrak{g}^*_{\mathrm{ad}})\big) = \mathrm{Td}(M_0)$. □

Proposition 4.2. *The formal equivariant cohomology class $S_\lambda(X)$ maps to $\mathcal{S}_\lambda(a_1, \ldots, a_{n(G)})$ under κ, where*

$$S_\lambda(X) = \frac{\sum_{\sigma\in W}(-1)^\sigma e^{(\sigma(\lambda+\rho),X)}}{\prod_{\gamma>0}\big(e^{\frac{1}{2}(\gamma,X)} - e^{-\frac{1}{2}(\gamma,X)}\big)}. \tag{24}$$

Proof: The expression defining S_λ is the same as the expression (11) defining $\mathcal{S}_\lambda$, with the expression $\mathcal{E}$ replaced by the variable $X \in \mathfrak{t}$. Thus $S_\lambda \in S(\mathfrak{t}^*)^W$ can be viewed as an equivariant cohomology class on M.

Since S_λ is symmetric under the action of the Weyl group, it is a function of the invariant polynomials $\tau_r(X)$. Since $\kappa(\tau_r) = a_r$ by definition of the a_r, when κ is applied to S_λ, the result will be $\mathcal{S}_\lambda$. Since κ is a ring homomorphism, the result follows. $\square$

The expression (22) is equal to the expression (10) which we derived in Section 2 for

$$\int_{M_0} e^{k\omega_0} \operatorname{Td}(M_0) \mathcal{S}_\lambda(a_1, \dots, a_{n(G)}). \tag{25}$$

To see this, we only need to evaluate (10) using the residue formula, and use the fact that

$$\frac{1}{\prod_{\gamma>0} \gamma(X)} \operatorname{Td}_G(\mathcal{O}_\Lambda)|_{F \times \{\sigma\Lambda\}} = (-1)^\sigma \prod_{\gamma>0} \frac{1}{1 - e^{-\sigma\gamma(X)}}$$

(which appears in (22)) is the same as

$$\frac{e^{\sigma\rho}}{\prod_{\gamma>0}(e^{\gamma/2} - e^{-\gamma/2})}$$

(which appears when we evaluate (10) using the residue formula). Hence (25) becomes

$$C \operatorname{res}\Big(\mathcal{D}^2(X) S_\lambda(X) \sum_{F \in \mathcal{F}} e^{i\mu_T(F)(X)} \int_F \frac{i_F^*\big(\operatorname{Td}_G(M) \operatorname{Td}_G^{-1}(\mathfrak{g}_{\mathrm{ad}} \oplus \mathfrak{g}_{\mathrm{ad}}^*) e^{k\omega}\big)}{e_F(X)} [dX]\Big), \tag{26}$$

which is the same as (22). Thus we have obtained a second proof of Theorem 2.3.

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